Back to the Roots - Polynomial System Solving Using Linear Algebra

Abstract

Polynomial system solving is a classical mathematical problem occurring in science and engineering. We return to the original algebraic roots of the problem of finding the solutions of a set of polynomial equations. Rather than approaching the problem from symbolic algebra, we review this task from the linear algebra perspective and show that interesting links with systems theory and realization theory emerge.

The system of polynomial equations is represented by a structured Macaulay coefficient matrix multiplied by a vector containing monomials. Two properties are of key importance in the null spaces of Macaulay coefficient matrices, namely the correspondence between linear (in)dependent monomials in the polynomials and the linear (in)dependent rows in the null space, and secondly, the occurrence of a monomial multiplication shift structure. Both properties are invariant and hence occur regardless of the specific numerical basis of the null space of the Macaulay matrix.

Based on these insights, two algorithms for finding the solutions of a system of multivariate polynomials are developed. The first algorithm proceeds by computing a basis for the null space of the Macaulay matrix. By exploiting the multiplication structure in the monomials, a generalized eigenvalue problem is derived in terms of matrices built up from certain rows of a numerically computed basis for the null space of the Macaulay matrix. The second procedure does not require the computation of a basis for the null space of the Macaulay matrix. Rather, it operates on certain columns of the Macaulay matrix and again employs the property that a set of monomials in the problem are linearly dependent on another set of monomials. By using a proper partitioning of the columns according to this separation into linearly independent monomials and linearly dependent monomials, the problem of finding the solutions is again formulated as an eigenvalue problem, in this case phrased using a certain partitioning of the Macaulay matrix. It is shown that this can be implemented in a numerically reliable manner using a (Q-less) QR decomposition. Furthermore, the generalization of the null space-based root-finding algorithm to the case of overconstrained systems is discussed. Several applications in system identification and computer vision are highlighted.

The developed solution methods bear a resemblance to the application of realization theory as encountered in systems theory and identification. We show that the null space of the Macaulay matrix can be interpreted as a state sequence matrix of an nD system realization. It turns out that the notions of the regular and singular parts of an nD descriptor system naturally correspond to the affine solutions and the solutions at infinity.